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B.M.S COLLEGE FOR WOMEN AUTONOMOUS

BENGALURU -560004

END SEMESTER EXAMINATION – APRIL/ MAY 2023

M.Sc. Mathematics – I Semester

REAL ANALYSIS

Course Code: MM102T **QP Code: 11002**

Duration: 3 Hours **Maximum Marks: 70**

Instructions: 1) **All** questions carry **equal** marks.

2) Answer **any five** full questions.

1. (a) Prove that $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$, for $\epsilon > 0$ however small, over the partition P on $[a,b]$.

   (b) If $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ then prove that $f_1 + f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$.

   (c) Show that $f(x) = -x$ is Riemann integrable on $[-k, 0], k$ is a positive constant.

   (5+5+4)

2. (a) If $f \in \mathcal{R}(\alpha)$ on $[a,b], m \leq f \leq M$ and ϕ is a continuous function on $[m, M]$, then show that $\phi(f) \in \mathcal{R}(\alpha)$ on $[a,b]$.

   (b) State and prove second mean value theorem. Further use it to evaluate

   $\int_0^2 (2x+5)\, d\{[x] - x\}$ **(7+7)**

3. (a) Define a total variation function. Calculate the total variation function of $f(x) = x - [x]$ on $[0,2]$, where $[x]$ is a greatest integer not exceeding x.

   (b) If f is bounded and integrable on $[a,b]$, then prove that there exists a function F defined as $F(x) = \int_a^x f(t)\, dt$, $a \leq x \leq b$ is continuous on $[a,b]$. Furthermore, if F is continuous at $x_0 \in [a,b]$, then show that F differentiable at x_0 and $F'(x) = f(x_0)$.

   (c) If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ then show that $f \in \mathcal{R}(c\alpha)$ on $[a,b]$, where c is a nonzero real constant. **(5+5+4)**

4. (a) State and prove Cauchy's principle for uniform convergence of sequence functions

$\{f_n(x)\}$ on $[a,b]$.

(b) Test for uniform convergence of $\{f_n(x)\} = \{x^{n-1}(1-x)\}$ on $[0,1]$.

(c) If a sequence of functions $\{f_n(x)\}$ converges to a function $f(x)$ defined on $[a,b]$ and $M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$. Then prove that $\{f_n(x)\}$ converges uniformly to $f(x)$ on $[a,b]$ if and only if $M_n \to 0$ $as\ n \to \infty$. **(6+4+4)**

5. **(**a) Let $\{f_n(x)\}$ be a sequence of differentiable functions on $[a,b]$ such that $\{f_n(x_0)\}$ converges for some x_0 on $[a,b]$. If the sequence $\{f_n'(x)\}$ converges uniformly on $[a,b]$, then show that $\{f_n(x)\}$ converges uniformly to a $f(x)$ on $[a,b]$ and $\lim_{n\to\infty} f_n'(x) = f'(x)$.

(b) If a series $\sum f_n(x)$ converges uniformly to a function $f(x)$ on $[a,b]$ and each $f_n(x) \in \mathcal{R}[a,b]$, then prove that $f(x) \in \mathcal{R}[a,b]$ and $\sum \int_a^b f_n(x)\ dx = \int_a^b f(x)\,dx$.

(c) Discuss the uniform convergence of $1 - x + x^2 - x^3 + x^4 - \cdots \ldots \ldots..; 0 < x < 1$.

(6+4+4)

6. If E is a subset of $\mathbb{R}^n$, then show that following statements are equivalent:

(i) E is closed and bounded

(ii) E is compact

(iii) Every infinite subset of E has a limit point in E. **(14)**

7. (a) Let $\vec{f}: E \to \mathbb{R}^m$ be differentiable where E be an open subset of $\mathbb{R}^n$, then prove that $D_j f_i$ exists for each $i = 1,2,\ldots..m$ and $j = 1,2,\ldots..n$. Further

$\overrightarrow{f'}(x)(e_j) = \sum_{i=1}^{m} (D_j f_i)(x) u_i$ where $\{u_1, u_2, \ldots..u_m\}$ is standard basis in $\mathbb{R}^m$.

(b) State and prove Banach fixed point theorem.

(c) Discuss the continuity of the function $\vec{f}: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & x \neq 0, y \neq 0 \\ 0, & x = 0, y = 0 \end{cases}$$ **(4+6+4)**

8. State and prove implicit function theorem. **(14)**

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