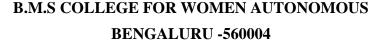
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END SEMESTER EXAMINATION – APRIL/ MAY 2023

M.Sc. Mathematics – I Semester REAL ANALYSIS

Course Code: MM102T Duration: 3 Hours

QP Code: 11002 Maximum Marks: 70

Instructions: 1) All questions carry equal marks.

2) Answer any five full questions.

- 1. (a) Prove that $f \in \mathcal{R}(\alpha)$ on [a,b] if and only if $U(P,f,\alpha) L(P,f,\alpha) < \epsilon$, for $\epsilon > 0$ however small, over the partition *P* on [a,b].
 - (b) If $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in \mathcal{R}(\alpha)$ on [a, b] then prove that $f_1 + f_2 \in \mathcal{R}(\alpha)$ on [a, b].
 - (c) Show that f(x) = -x is Riemann integrable on [-k, 0], k is a positive constant.

(5+5+4)

- 2. (a) If $f \in \mathcal{R}(\alpha)$ on $[a, b], m \le f \le M$ and ϕ is a continuous function on [m, M], then show that $\phi(f) \in \mathcal{R}(\alpha)$ on [a, b].
 - (b) State and prove second mean value theorem. Further use it to evaluate

$$\int_0^2 (2x+5) d\{[x]-x\}$$
(7+7)

- 3. (a) Define a total variation function. Calculate the total variation function of f(x) = x [x] on [0,2], where [x] is a greatest integer not exceeding x.
 - (b) If f is bounded and integrable on [a, b], then prove that there exists a function F defined as F(x) = ∫_a^x f(t) dt, a ≤ x ≤ b is continuous on [a, b]. Furthermore, if F is continuous at x₀ ∈ [a, b], then show that F differentiable at x₀ and F'(x) = f(x₀).
 - (c) If $f \in \mathcal{R}(\alpha)$ on [a, b] then show that $f \in \mathcal{R}(c\alpha)$ on [a, b], where c is a nonzero real constant. (5+5+4)
- 4. (a) State and prove Cauchy's principle for uniform convergence of sequence functions

 $\{f_n(x)\}$ on [a, b].

- (b) Test for uniform convergence of $\{f_n(x)\} = \{x^{n-1}(1-x)\}$ on [0,1].
- (c) If a sequence of functions $\{f_n(x)\}$ converges to a function f(x) defined on [a, b] and $M_n = \sup_{x \in [a,b]} |f_n(x) - f(x)|$. Then prove that $\{f_n(x)\}$ converges uniformly to
 - f(x) on [a, b] if and only if $M_n \to 0$ as $n \to \infty$. (6+4+4)
- 5. (a) Let $\{f_n(x)\}\$ be a sequence of differentiable functions on [a, b] such that $\{f_n(x_0)\}\$ converges for some x_0 on [a, b]. If the sequence $\{f_n'(x)\}$ converges uniformly on [a, b], then show that $\{f_n(x)\}\$ converges uniformly to a f(x) on [a, b] and $\lim_{n\to\infty} f'_n(x) = f'(x)$.
 - (b) If a series $\sum f_n(x)$ converges uniformly to a function f(x) on [a, b] and each

 $f_n(x) \in \mathcal{R}[a, b]$, then prove that $f(x) \in \mathcal{R}[a, b]$ and $\sum \int_a^b f_n(x) dx = \int_a^b f(x) dx$.

(c) Discuss the uniform convergence of $1 - x + x^2 - x^3 + x^4 - \dots \dots; 0 < x < 1.$ (6+4+4

(6+4+4)

(14)

- If E is a subset of \mathbb{R}^n , then show that following statements are equivalent: 6.
 - E is closed and bounded (i)
 - E is compact (ii)
 - (iii) Every infinite subset of E has a limit point in E. (14)
- 7. (a) Let $\vec{f}: E \to \mathbb{R}^m$ be differentiable where *E* be an open subset of \mathbb{R}^n , then prove that $D_j f_i$ exists for each i = 1, 2, ..., m and j = 1, 2, ..., n. Further

$$\vec{f'}(x)(e_j) = \sum_{i=1}^m (D_j f_i)(x) u_i$$
 where $\{u_1, u_2, \dots, u_m\}$ is standard basis in \mathbb{R}^m .

- (b) State and prove Banach fixed point theorem.
- (c) Discuss the continuity of the function $\vec{f}: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2 + y^2}, & x \neq 0, y \neq 0\\ 0, & x = 0, y = 0 \end{cases}$$
(4+6+4)

State and prove implicit function theorem. 8.
